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# ON THE VIBRATIONS OF AN *N*-STRING

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We study the small-amplitude transverse oscillations of a planar network of $N$ sections of string which are all attached at one common extremity. This network is called an $N$-string. When the $N$ sections of string are of finite length, we find Fourier series expressions describing the vibrations perpendicular to the plane containing the $N$-string at rest. The standing perpendicular wave energies of a plucked symmetric $N$-string are analyzed. It is found that higher harmonics can be excited to an energy level above that of the first harmonic simply by plucking at an appropriate location along one of the strings. This result is in contrast to an ordinary plucked string and may lead to interesting applications; most notably the construction of new musical instruments. We also describe the movements of one travelling perpendicular wave in an $N$-string as well as the interaction of such waves. A method for increasing or reducing the amplitude of travelling perpendicular waves is outlined.



## 1. INTRODUCTION

The vibrating string problem, in which a tightly stretched flexible string is initially perturbed and then allowed to vibrate, remains one of the important problems of mathematical physics [1] and acoustics [2]. Also, because of its simplicity, it has become a classical problem in the theory of partial differential equations [3]. In this work we consider an extension of the vibrating string problem.

We define an $N$-string as a planar network of $N$ strings that are connected at one common extremity. The common extremity, or the junction point, is mobile while the other extremities are all fixed (see Figure 1). The aim of this paper is to study the linearized vibrations of an $N$-string having finite or infinite length strings. We will obtain Fourier- and d'Alembert-type expressions describing the vibrations which are perpendicular to the plane containing the $N$-string at rest. These solutions for the linearized problem are useful in two regards: (1) they describe the small-amplitude perpendicular oscillations of the $N$-string, and (2) knowledge of the eigenmodes and eigenvalues of this linear problem is essential for any reduction method (for example, center manifold or Lyapunov–Schmidt reduction, see reference [4]) which would be used to study the stability and bifurcations of the rest state of the $N$-string in a non-linear model.

The analysis of the small-amplitude vibrations of an $N$-string is interesting from an acoustic point of view. Let us first recall that all string instruments produce sounds that are

 

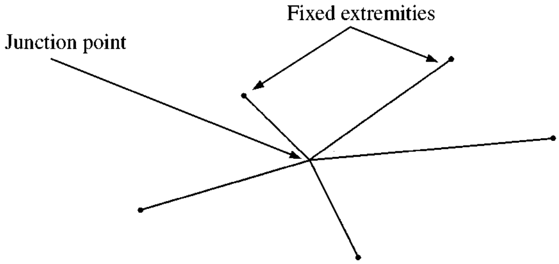


Figure 1. Schematic of a 5-string. All the extremeties are fixed while the junction point is free to move.

well described through the analysis of small-amplitude vibrations of an ordinary string which is plucked or struck. The sound produced by each string of such instruments can be decomposed into modes which have energies that are always dominated by the energy of its fundamental mode. If the string is plucked at its midpoint, the energy of its modes are in harmonic relation with the energy of its fundamental mode. In this case, over 80 per cent of the total energy is in the fundamental mode. One of the goals of this paper is to show that the relationships between the energies of the modes of an $N$-string are significantly different from the case of an ordinary string. In fact, we will show that higher modes of an $N$-string, $N > 2$, can be excited to energy levels above that of the fundamental mode, simply by plucking at an appropriate location along one of the strings which compose a symmetric $N$-string. We will also show that when such an $N$-string is plucked at the junction point of its strings, the energy levels of its modes are in the same harmonic relations as in the case of an ordinary string. These characteristics of the $N$-string should give instruments, made up of $N$-strings, unique acoustic features.

The paper is organized as follows. In section 2, we present our mathematical model and show the uncoupling of planar and perpendicular waves of an $N$-string at the linear level. We establish some theoretical results and study the vibrations of standing perpendicular waves of an $N$-string in section 3. In section 4, we analyze the standing perpendicular wave energies of a symmetric $N$-string. In section 5, we consider the vibrations of an $N$-string having all but one identical strings. We describe the motion of one travelling perpendicular wave in an $N$-string in section 6. Section 7 deals with interactions of travelling perpendicular waves first in one $N$-string and second in a tree of $N$-strings. Section 8 contains the conclusion and a discussion of our results.

## 2. LINEARIZED MATHEMATICAL MODEL

In the mathematical treatment that follows, we make the following assumptions about the $N$-string:

1. Each string of the $N$-string is flexible and elastic.
2. The elasticity of each string of the $N$-string satisfies Hooke's law.
3. The $N$-string is subject to no internal nor external friction.
4. At rest, the $N$-string is in a plane.

Using a variational principle, Schmidt [5] presents a non-linear system of partial differential equations with appropriate boundary and coupling conditions which describe the planar vibrations of an arbitrary planar network of strings. He then derives and studies the linearization of this model about the rest position of the network.

Without difficulty, one can extend Schmidt's approach to describe the full three-dimensional motion of a planar network of strings and the corresponding linearized model. Specifically, let $x_i$, $i = 1, 2, \ldots, N$, be the arclength parameter for the $i$th string at rest, whose "natural" length is assumed to be $l_i$, i.e., $0 \leqslant x_i \leqslant l_i$, where we assume that the common junction point of the strings is at $x_i = 0$, $i = 1, \ldots, N$. Also, let $\mathbf{r}^i(x_i, t)$ denote the three-dimensional vector which gives the deviation from the rest position at time $t$ of the point at arclength position $x_i$ on the $i$th string. We can assume, without loss of generality, that the first two components of $\mathbf{r}^i$ give the motions in the plane of the $N$-string and that the third component of $\mathbf{r}^i$ gives the motions perpendicular to the plane of the $N$-string. Then following Schmidt [5], the linearized boundary value problem for the $\mathbf{r}^i$ is the following:

$$\mathbf{r}^i_{tt} = \Gamma_i \mathbf{r}^i_{x_i x_i}, \quad i = 1, 2, \ldots, N, \tag{1}$$

$$\mathbf{r}^1(0, t) = \mathbf{r}^2(0, t) = \cdots = \mathbf{r}^N(0, t), \tag{2}$$

$$\mathbf{r}^i(l_i, t) = 0, \quad i = 1, 2, \ldots, N, \tag{3}$$

$$\sum_{i=1}^{N} \rho_i \Gamma_i \mathbf{r}^i_{x_i}(0, t) = 0, \tag{4}$$

where $\Gamma_i$ is a symmetric block-diagonal $3 \times 3$ matrix of the form

$$\Gamma_i = \begin{pmatrix} \gamma_{i1} & \gamma_{i2} & 0 \\ \gamma_{i2} & \gamma_{i3} & 0 \\ 0 & 0 & c_i^2 \end{pmatrix},$$

where $c_i > 0$, $\gamma_{i1}$, $\gamma_{i2}$ and $\gamma_{i3}$ are constants depending on the tension and the Hooke's law constant for the $i$th string, as well as on the mass density $\rho_i > 0$ of the $i$th string. Consequently, equations (1)–(4) uncouple into planar components and a perpendicular component. Note that this uncoupling does not occur for the full non-linear model [4]. If we designate by $u_i$ the third component of the vector $\mathbf{r}^i$, then equations (1)–(4) lead to the following equations:

$$u^i_{tt} = c_i^2 u^i_{x_i x_i}, \quad i = 1, 2, \ldots, N, \tag{5}$$

$$u^1(0, t) = u^2(0, t) = \cdots = u^N(0, t), \tag{6}$$

$$u^i(l_i, t) = 0, \quad i = 1, 2, \ldots, N, \tag{7}$$

$$\sum_{i=1}^{N} c_i^2 \rho_i u^i_{x_i}(0, t) = 0, \tag{8}$$

We wish to solve the problem (5)–(8) subject to the initial conditions

$$u^i(x_i, 0) = F^i(x_i), \quad u^i_t(x_i, 0) = G^i(x_i), \quad i = 1,2, \ldots, N, \tag{9}$$

where the functions $F^i$ and $G^i$ are respectively continuous and piecewise continuous on $[0, l_i]$.

## 3. STANDING WAVES

In this section as well as in sections 4 and 5, we will assume that all strings of the $N$-string have finite lengths.

We will solve the problem (5)–(9) by the method of separation of variables, which will lead to a matrix Sturm–Liouville system. Therefore, we will begin our study by giving some information relevant to our problem on the eigenfunctions and eigenvalues of this kind of Sturm–Liouville system.

### 3.1. STURM–LIOUVILLE PROBLEM

A normalization of the strings will facilitate our work. Let $x := \pi x_i / l_i$ for $i = 1, 2, \ldots, N$. It follows that $0 \leqslant x \leqslant \pi$ parameterizes each string. If one defines $v^i(x, t) := u^i(l_i x/\pi, t)$ for $i = 1, 2, \ldots, N$, then the problem (5)–(9) becomes

$$v^i_{tt} = \frac{\pi^2 c_i^2}{l_i^2} v^i_{xx}, \quad i = 1, 2, \ldots, N, \tag{10}$$

$$v^1(0, t) = v^2(0, t) = \cdots = v^N(0, t), \tag{11}$$

$$v^i(\pi, t) = 0, \quad i = 1, 2, \ldots, N, \tag{12}$$

$$\sum_{i=1}^{N} \frac{c_i^2 \rho_i}{l_i} v^i_x(0, t) = 0, \tag{13}$$

$$v^i(x, 0) = F^i(l_i x/\pi), \quad v^i_t(x, 0) = G^i(l_i x/\pi), \quad i = 1,2, \ldots, N. \tag{14}$$

Let $C := \operatorname{diag}(\pi^2 c_1^2/l_1^2, \pi^2 c_2^2/l_2^2, \ldots, \pi^2 c_N^2/l_N^2)$ and $L := \operatorname{diag}(l_1\rho_1, l_2\rho_2, \ldots, l_N\rho_N)$. Let us also designate by $\langle\, , \rangle_L$ the scalar product on $\mathbf{R}^N$ obtained from the matrix $L$. Hence, if $\Phi := (\phi_1, \phi_2, \ldots, \phi_N)^{\mathrm{T}}$, $\Psi := (\psi_1, \psi_2, \ldots, \psi_N)^{\mathrm{T}} \in \mathbf{R}^N$, then

$$\langle \Phi, \Psi \rangle_L = l_1\rho_1\phi_1\psi_1 + l_2\rho_2\phi_2\psi_2 + \cdots + l_N\rho_N\phi_N\psi_N.$$

Note that since $CL = LC$ then $\langle C\Phi, \Psi \rangle_L = \langle \Phi, C\Psi \rangle_L$.

Let $\mathbf{L}^2[0, \pi]$ and $\mathbf{W}^1[0, \pi]$ be the usual Hilbert and Sobolev spaces on $[0, \pi]$, and let us denote $\mathbf{\Lambda} := (\mathbf{L}^2[0, \pi])^N$ and $\mathbf{\Omega} := (\mathbf{W}^1[0, \pi])^N$, i.e., the set of $N$-dimensional vector-valued functions whose components are respectively in $\mathbf{L}^2[0, \pi]$ and $\mathbf{W}^1[0, \pi]$. We will use the following scalar product, induced from $\langle\, , \rangle_L$, on the Hilbert space $\mathbf{\Lambda}$:

$$\langle\langle \Phi, \Psi \rangle\rangle_L := \int_0^\pi \langle \Phi(x), \Psi(x) \rangle_L \, \mathrm{d}x.$$

Let

$$\mathbf{\Delta} := \{\Phi \in \mathbf{\Omega} : \phi^1(0) = \phi^2(0) = \cdots = \phi^N(0), \Phi(\pi) = 0, \langle C\Phi'(0), (1, 1, \ldots, 1)^{\mathrm{T}} \rangle_L = 0\}.$$

We consider the operator $A : \mathbf{\Delta} \to \mathbf{\Lambda}$ defined by $A\Phi = -C\Phi''$. It can be shown that the operator $A$ is symmetric with respect to $\langle\langle\, , \rangle\rangle_L$ on $\mathbf{\Delta}$. It then follows that the eigenvalues of $A$ are real numbers and if $\Psi_1$ and $\Psi_2$ are two eigenfunctions of $A$ corresponding respectively to the eigenvalues $\omega_1$ and $\omega_2$, $\omega_1 \neq \omega_2$, then $\langle\langle \Psi_1, \Psi_2 \rangle\rangle_L = 0$. We also have that the spectrum of $A$ is composed only of discrete eigenvalues of multiplicity at most $2N$ and the eigenvalues of $A$ constitute a real monotone sequence $\{\omega_n\}$ such that

$$\omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_n \leqslant \omega_{n+1} \leqslant \cdots, \lim_{n\to\infty} \omega_n = \infty.$$

Finally, we have that there exists an orthonormal set $\{\Phi_i\}$ of eigenfunctions of $A$ which is complete in $\mathbf{\Lambda}$ (see Appendix A for proof).

### 3.2. SOLVING THE GENERAL CASE

In the previous section it was shown that the eigenfunctions of $A$ form a basis for $\mathbf{\Lambda}$. We will now show, using the method of separation of variables, that this basis is the "natural" basis in which to study (10)–(14). We therefore set

$$v^i(x, t) = X^i(x)T(t), \quad i = 1, 2, \dots, N, \tag{15}$$

where the $X^i$ are functions of $x$ only and $T$ is a function of $t$ only. Substitution of equation (15) into equations (10)–(14) produces

$$T'' + \omega T = 0, \tag{16}$$

$$CX'' + \omega X = 0, \tag{17}$$

$$X^1(0) = X^2(0) = \cdots = X^N(0), \tag{18}$$

$$X(\pi) = 0, \tag{19}$$

$$\langle CX'(0), (1, 1, \dots, 1)^{\mathrm{T}} \rangle_L = 0, \tag{20}$$

where $X := (X^1, X^2, \dots, X^N)^{\mathrm{T}}$ and $-\omega$ is the separation constant. It is straightforward to show that equations (17)–(20) has non-trivial solutions only if $\omega > 0$. Thus, we can set $\lambda := \sqrt{\omega} > 0$. Consequently, equation (16) yields

$$T(t) = K_1 \cos \lambda t + K_2 \sin \lambda t,$$

where $K_1$ and $K_2$ are arbitrary constants. We also have that $X$ is an eigenfunction of $A$ with eigenvalue $\lambda^2$. The solution of equation (17) is

$$X^i(x) = A^i \cos \frac{\lambda l_i x}{\pi c_i} + B^i \sin \frac{\lambda l_i x}{\pi c_i}, \quad i = 1, 2, \dots, N,$$

where the $A^i$ and $B^i$ are arbitrary constants. Conditions (18) leads to

$$A^1 = A^2 = \cdots = A^N, \tag{21}$$

while condition (20) gives

$$\sum_{i=1}^{N} v_i B^i = 0, \tag{22}$$

where we define $v_i := c_i \rho_i$. Finally, equations (19), (21) and (22) lead to the following system of equations:

$$\begin{aligned} &A^1 \cos \tfrac{l_1 \lambda}{c_1} - \tfrac{1}{v_1}(B^2 v_2 + \cdots + B^N v_N) \sin \tfrac{l_1 \lambda}{c_1} = 0, \\ &A^1 \cos \tfrac{l_i \lambda}{c_i} + B^i \sin \tfrac{l_i \lambda}{c_i} = 0, \quad i = 2, 3, \dots, N. \end{aligned} \tag{23}$$

Thus, $\lambda^2$ is an eigenvalue of the operator $A$ if and only if $\lambda$ is such that equation (23) has a non-trivial solution for $A^1, B^2, \ldots, B^N$. It is easy to see that this occurs if and only if $\lambda$ is such that

$$\sum_{i=1}^{N}\left[\frac{v_i}{v_1}\cos\left(\frac{l_i\lambda}{c_i}\right)\prod_{\substack{j=1\\ j\neq i}}^{N}\sin\left(\frac{l_j\lambda}{c_j}\right)\right]=0. \tag{24}$$

Equation (24) has an infinite number of roots $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n\leqslant\lambda_{n+1}\leqslant\cdots$, with $\lim_{n\to\infty}\lambda_n=\infty$. Note that the actual values of these roots depend on the values of the $v_i$, $l_i$ and $c_i$.

Let $\lambda$ be a solution to equation (24). There are two possibilities: either $\sin(l_i\lambda/c_i)\neq 0$ for all $i=1,2,\ldots,N$, or there exists an $i_1\in\{1,2,\ldots,N\}$ such that $\sin(l_{i_1}\lambda/c_{i_1})=0$. It turns out that these two cases are intimately linked to whether or not the real numbers $c_i/l_i$, $i=1,2,\ldots,N$ are incommensurate. We study each case separately below.

### 3.2.1. *Incommensurate case*

Suppose the numbers $c_i/l_i$, $i=1,2,\ldots,N$, are incommensurate, i.e., $c_il_j/c_jl_i\notin\mathbf{Q}$, for all $i,j=1,2,\ldots,N$, $i\neq j$. Also suppose $\lambda$ is a solution of equation (24) such that there exists an $i_1\in\{1,2,\ldots,N\}$ with $\sin\{l_{i_1}\lambda/c_{i_1})=0$. Then clearly $\lambda=n_1\pi c_{i_1}/l_{i_1}$ for some positive integer $n_1$. Equation (24) in this case reduces to

$$\frac{v_{i_1}}{v_1}\cos n_1\pi\prod_{\substack{j=1\\ j\neq i_1}}^{N}\sin\left(\frac{c_{i_1}l_j}{c_jl_{i_1}}n_1\pi\right)=0,$$

from which it follows that there exists a $i_2\in\{1,2,\ldots,N\}$, $i_2\neq i_1$, and a positive integer $n_2=c_{i_1}l_{i_2}n_1/c_{i_2}l_{i_1}$. The numbers $c_{i_1}/l_{i_1}$ and $c_{i_2}/l_{i_2}$ would thus be commensurate, which is a contradiction. This means that, in this case, every solution $\lambda$ of equation (24) is such that $\sin(l_i\lambda/c_i)\neq 0$ for $i=1,2,\ldots,N$. Consequently, if the $c_i/l_i, i=1,2,\ldots,N$, are incommensurate, equation (24) has an infinite number of solutions $\lambda_k=\alpha_k$, $k\in\mathbf{N}^*$, and each $\alpha_k$ is such that $\sin(l_i\alpha_k/c_i)\neq 0$ for $i=1,2,\ldots,N$. The corresponding eigenvalue is simple and the eigenfunction is given by

$$\begin{aligned}P_k(x):=\Bigg[&\cos\frac{l_1\alpha_kx}{\pi c_1}+\left(\frac{v_2}{v_1}\cot\frac{l_2\alpha_k}{c_2}+\cdots+\frac{v_N}{v_1}\cot\frac{l_N\alpha_k}{c_N}\right)\sin\frac{l_1\alpha_kx}{\pi c_1},\\ &\cos\frac{l_2\alpha_kx}{\pi c_2}-\left(\cot\frac{l_2\alpha_k}{c_2}\right)\sin\frac{l_2\alpha_kx}{\pi c_2},\ldots,\cos\frac{l_N\alpha_kx}{\pi c_N}-\left(\cot\frac{l_N\pi}{c_N}\right)\sin\frac{l_N\alpha_kx}{\pi c_N}\Bigg]^{\mathrm{T}}.\end{aligned} \tag{25}$$

Hence, for any $\Phi$ in the Hilbert space $\mathbf{\Lambda}$, we have

$$\Phi(x)=\sum_{k=1}^{\infty}A_kP_k(x),$$

where

$$A_k:=\frac{\langle\langle\Phi,P_k\rangle\rangle_L}{\langle\langle P_k,P_k\rangle\rangle_L}.$$

Therefore, the solution (possibly a weak solution) of equations (10)–(14), where the numbers $c_i/l_i$, $i = 1, 2, \ldots, N$, are incommensurate, can be written as

$$[v^1(x, t), v^2(x, t), \ldots, v^N(x, t)]^{\mathrm{T}} = \sum_{k=1}^{\infty} (a_k \cos \alpha_k t + \hat{a}_k \sin \alpha_k t) P_k(x),$$

where

$$a_k := \frac{\langle\langle F, P_k \rangle\rangle_L}{\langle\langle P_k, P_k \rangle\rangle_L}, \quad \hat{a}_k := \frac{\langle\langle G, P_k \rangle\rangle_L}{\alpha_k \langle\langle P_k, P_k \rangle\rangle_L}$$

for

$$F(x) := [F^1(l_1 x/\pi), F^2(l_2 x/\pi), \ldots, F^N(l_N x/\pi)]^{\mathrm{T}},$$

$$G(x) := [G^1(l_1 x/\pi), F^2(l_2 x/\pi), \ldots, G^N(l_N x/\pi)]^{\mathrm{T}}.$$

The solution of the problem (5)–(9) is finally obtained through the substitution

$$u^i(x_i, t) = v^i(\pi x_i / l_i, t), \quad i = 1, 2, \ldots, N.$$

### 3.2.2. *Commensurate case*

The case where some of the $c_i/l_i$ are commensurate requires special care. A few new definitions are in order. Let $\mu_1, \mu_2, \ldots, \mu_N$ be positive real numbers. We will call resonance† of the positive numbers $\mu_1, \ldots, \mu_N$ a relation of the form

$$n_1 \mu_{i_1} = n_2 \mu_{i_2} = \cdots = n_r \mu_{i_r}, \quad r \in \{2, 3, \ldots, N\}, \tag{26}$$

where

- $i_1, i_2, \ldots, i_r$ are distinct elements of $\{1, 2, \ldots, N\}$,
- $n_1, n_2, \ldots, n_r$ are positive integers without common divisor larger than 1, and
- the set $\{\mu_{i_1}, \ldots, \mu_{i_r}\}$ is maximal in the sense that there exists no integer $k$ such that $n_1 \mu_{i_1} = k \mu_j$ for some $j \in \{1, 2, \ldots, N\} \backslash \{i_1, i_2, \ldots, i_r\}$.

If $s > r$, we will say that

$$n_1 \mu_{i_1} = n_2 \mu_{i_2} = \cdots = n_r \mu_{i_r}$$

is a subresonance of the resonance

$$m_1 \mu_{j_1} = m_2 \mu_{j_2} = \cdots = m_s \mu_{j_s}$$

if $\{\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_r}\}$ is a proper subset of $\{\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_s}\}$. In this case, there exists a number $K \in \mathbf{N}, K > 1$, such that if $\mu_{i_a} = \mu_{j_b}$ for $i_a \in \{i_1, i_2, \ldots, i_r\}$ and $j_b \in \{j_1, j_2, \ldots, j_s\}$, then $m_b = K n_a$. We will say that $K$ is the order of inclusion of the subresonance in the resonance. Finally, we will say that a resonance is maximal if it is not a subresonance of any other resonance.

† This use of the term resonance should not be confused with the terminology used in the theory of Poincaré normal forms of non-linear ordinary differential equations. For our purposes, we will need the full chain of equations in equation (26). If one considers just the individual equalities, and if the $\mu_i$ were representing the fundamental frequencies of the movement, then the $\mu_i$ would form, two by two, resonances in the usual sense.

According to section 3.2.1 above, we know that if $\lambda$ is a solution to equation (24) such that there exists an $i \in \{1, 2, \dots, N\}$ with $\sin(l_i\lambda/c_i) = 0$, then there must be a resonance of the real numbers

$$\frac{c_1}{l_1}, \frac{c_2}{l_2}, \dots, \frac{c_N}{l_N}.$$

In general, there will be $q$ such resonances which we write as

$$\begin{aligned} n_{1,1}\frac{c_{i_{1,1}}}{l_{i_{1,1}}} &= n_{1,2}\frac{c_{i_{1,2}}}{l_{i_{1,2}}} = \cdots = n_{1,p_1}\frac{c_{i_1,p_1}}{l_{i_1,p_1}}, \\ n_{2,1}\frac{c_{i_{2,1}}}{l_{i_{2,1}}} &= n_{2,2}\frac{c_{i_{2,2}}}{l_{i_{2,2}}} = \cdots = n_{2,p_2}\frac{c_{i_2,p_2}}{l_{i_2,p_2}}, \\ &\vdots \\ n_{q,1}\frac{c_{i_{q,1}}}{l_{i_{q,1}}} &= n_{q,2}\frac{c_{i_{q,2}}}{l_{i_{q,2}}} = \cdots = n_{q,p_q}\frac{c_{i_q,p_q}}{l_{i_q,p_q}}, \end{aligned} \tag{27}$$

where $2 \leqslant p_k \leqslant N$ for $k = 1, 2, \dots, q$. We can assume without loss of generality that $i_{k,1} < i_{k,2} < \cdots < i_{k,p_k}$ for each $k = 1, 2, \dots, q$, and there exists an integer $M$, $1 \leqslant M \leqslant q$, such that the $M$ first resonances in (27) are maximal, and the remaining resonances (if any) are not maximal.

Let us consider the $j$th maximal resonance in (27), i.e., $1 \leqslant j \leqslant M$. For any positive integer $k$, we have that $\lambda_k = kn_{j,1}\pi c_{i_{j,1}}/l_{i_{j,1}}$ is a solution of equation (24), and there are $p_j - 1$ orthogonal (with respect to $\langle\langle\, , \rangle\rangle_L$) eigenfunctions for each value of $k$; these eigenfunctions will be designated by $Q_{j,k,m}(x)$ for $1 \leqslant m \leqslant p_j - 1$. The components of $Q_{j,k,m}(x)$ are as follows:

$$Q^i_{j,k,m}(x) = 0, \quad \text{if } i \notin \{i_{j,1}, i_{j,2}, \dots, i_{j,p_j}\}$$

and

$$Q_{j,k,1}(x): \begin{cases} Q^{i_{j,1}}_{j,k,1}(x) = \left[\dfrac{1}{v_{i_{j,1}}}\left(\dfrac{v_{i_{j,2}}}{n_{j,2}} + \dfrac{v_{i_{j,3}}}{n_{j,3}} + \cdots + \dfrac{v_{i_{j,p_j}}}{n_{j,p_j}}\right)\right] \sin kn_{j,1}x, \\ Q^{i_{j,2}}_{j,k,1}(x) = \dfrac{-1}{n_{j,2}} \sin kn_{j,2}x, \\ \vdots \\ Q^{i_{j,p_j}}_{j,k,1}(x) = \dfrac{-1}{n_{j,p_j}} \sin kn_{j,p_j}x, \end{cases}$$

$$Q_{j,k,2}(x): \begin{cases} Q^{i_{j,1}}_{j,k,2}(x) = 0, \\ Q^{i_{j,2}}_{j,k,2}(x) = \left[\dfrac{1}{v_{i_{j,2}}}\left(\dfrac{v_{i_{j,3}}}{n_{j,3}} + \cdots + \dfrac{v_{i_{j,p_j}}}{n_{j,p_j}}\right)\right] \sin kn_{j,2}x, \\ Q^{i_{j,3}}_{j,k,2}(x) = \dfrac{-1}{n_{j,3}} \sin kn_{j,3}x, \\ \vdots \\ Q^{i_{j,p_j}}_{j,k,2}(x) = \dfrac{-1}{n_{j,p_j}} \sin kn_{j,p_j}x, \end{cases}$$

$$\vdots$$

$$Q_{j,k,p_j-1}(x) : \begin{cases} Q^{i_{j,1}}_{j,k,p_j-1}(x) = Q^{i_{j,2}}_{j,k,p_j-1}(x) = \cdots = Q^{i_{j,p_j-2}}_{j,k,p_j-1}(x) = 0, \\ Q^{i_{j,p_j-1}}_{j,k,p_j-1}(x) = \left[\dfrac{1}{v_{i_{j,p_j-1}}}\left(\dfrac{v_{i_{j,p_j}}}{n_{j,p_j}}\right)\right] \sin kn_{j,p_j-1}x, \\ Q^{i_{j,p_j}}_{j,k,p_j-1}(x) = \dfrac{-1}{n_{j,p_j}} \sin kn_{j,p_j}x. \end{cases}$$

Now, suppose that $M < q$, i.e., there is at least one resonance in equation (27) which is not maximal. If $\tilde{j}$ is an integer such that $M < \tilde{j} \leqslant q$, then there exists a set $\{K_1, K_2, \ldots, K_\omega\}$ of orders of inclusion of the $\tilde{j}$th subresonance into the other resonances. We still have that $\lambda_k = kn_{\tilde{j},1}\pi c_{i_{\tilde{j},1}}/l_{i_{\tilde{j},1}}$ is a solution to equation (24) for any integer $k$; however, if $k$ is divisible by any one of the integers $K_1, \ldots, K_\omega$, then this eigenvalue is already accounted for in a resonance which contains the $\tilde{j}$th resonance in equation (27) as a subresonance. Hence, we only consider the integers $\tilde{k}$ which are not a multiple of any of $K_1, \ldots, K_\omega$. The corresponding orthogonal eigenfunctions $\tilde{Q}_{\tilde{j},\tilde{k},m}(x)$ are given by the same expressions as the $Q_{j,k,m}(x)$ previously given.

In addition to the eigenfunctions $Q_{j,k,m}(x)$ and $\tilde{Q}_{\tilde{j},\tilde{k},m}(x)$, eigenfunctions of the form (25) are also present when the $c_i/l_i$, $i = 1, 2, \ldots, N$, are commensurate. In fact, suppose there are $q$ resonances given by equation (27). Then there exists an infinite number of solutions to (24), $\lambda_k = \alpha_k$, $k \in \mathbf{N}^*$, for which $\sin(l_i\alpha_k/c_i) \neq 0$, $i = 1, 2, \ldots, N$. The corresponding eigenfunctions are given by equation (25) (see Appendix B for the proof).

Hence, if $\Phi$ is any element of $\mathbf{\Lambda}$, we can write

$$\Phi(x) = \sum_{k=1}^{\infty} A_k P_k(x) + \sum_{j=1}^{M} \sum_{m=1}^{p_j-1} \sum_{k=1}^{\infty} B_{j,k,m} Q_{j,k,m}(x) + \sum_{\tilde{j}=M+1}^{q} \sum_{m=1}^{p_{\tilde{j}}-1} \sum_{\tilde{k}} C_{\tilde{j},\tilde{k},m} \tilde{Q}_{\tilde{j},\tilde{k},m}(x),$$

with the understanding that if $M = q$, then the third sum in the above formula does not appear. It is also understood that in this third sum, $\tilde{k}$ only takes on restricted integer values as previously described. The coefficients are given by

$$A_k := \frac{\langle\langle \Phi, P_k \rangle\rangle_L}{\langle\langle P_k, P_k \rangle\rangle_L}, \quad B_{j,k,m} := \frac{\langle\langle \Phi, Q_{j,k,m} \rangle\rangle_L}{\langle\langle Q_{j,k,m}, Q_{j,k,m} \rangle\rangle_L}, \quad C_{\tilde{j},\tilde{k},m} := \frac{\langle\langle \Phi, \tilde{Q}_{\tilde{j},\tilde{k},m} \rangle\rangle_L}{\langle\langle \tilde{Q}_{\tilde{j},\tilde{k},m}, \tilde{Q}_{\tilde{j},\tilde{k},m} \rangle\rangle_L}.$$

We have thus shown that if there are $q$ resonances of the numbers $c_i/l_i$, $i = 1, 2, \ldots, N$, given by equation (27) (the first $M$ being maximal), the solution (possibly a weak solution) of equations (10)–(14) can be written as

$$\begin{aligned} &[v^1(x,t), v^2(x,t), \ldots, v^N(x,t)]^{\mathrm{T}} \\ &= \sum_{k=1}^{\infty} (a_k \cos \alpha_k t + \hat{a}_k \sin \alpha_k t) P_k(x) \\ &\quad + \sum_{j=1}^{M} \sum_{m=1}^{p_j-1} \sum_{k=1}^{\infty} [(b_{j,k,m} \cos(kn_{j,1}\pi c_{i_{j,1}} t/l_{i_{j,1}}) + \hat{b}_{j,k,m} \sin(kn_{j,1}\pi c_{i_{j,1}} t/l_{i_{j,1}})] Q_{j,k,m}(x) \\ &\quad + \sum_{\tilde{j}=M+1}^{q} \sum_{m=1}^{p_{\tilde{j}}-1} \sum_{\tilde{k}} [c_{\tilde{j},\tilde{k},m} \cos(\tilde{k}n_{\tilde{j},1}\pi c_{i_{\tilde{j},1}} t/l_{i_{\tilde{j},1}}) + \hat{c}_{\tilde{j},\tilde{k},m} \sin(\tilde{k}n_{\tilde{j},1}\pi c_{i_{\tilde{j},1}} t/l_{i_{\tilde{j},1}})] \tilde{Q}_{\tilde{j},\tilde{k},m}(x), \end{aligned}$$

where

$$a_k := \frac{\langle\langle F, P_k \rangle\rangle_L}{\langle\langle P_k, P_k \rangle\rangle_L}, \quad \hat{a}_k := \frac{\langle\langle G, P_k \rangle\rangle_L}{\alpha_k \langle\langle P_k, P_k \rangle\rangle_L},$$

$$b_{j,k,m} := \frac{\langle\langle F, Q_{j,k,m} \rangle\rangle_L}{\langle\langle Q_{j,k,m}, Q_{j,k,m} \rangle\rangle_L}, \quad \hat{b}_{j,k,m} := \frac{l_{i_{j,1}} \langle\langle G, Q_{j,k,m} \rangle\rangle_L}{k n_{j,1} c_{i_{j,1}} \pi \langle\langle Q_{j,k,m}, Q_{j,k,m} \rangle\rangle_L},$$

$$c_{\tilde{j},\tilde{k},m} := \frac{\langle\langle F, \tilde{Q}_{\tilde{j},\tilde{k},m} \rangle\rangle_L}{\langle\langle \tilde{Q}_{\tilde{j},\tilde{k},m}, \tilde{Q}_{\tilde{j},\tilde{k},m} \rangle\rangle_L}, \quad c_{\tilde{j},\tilde{k},m} := \frac{l_{i_{\tilde{j},1}} \langle\langle G, \tilde{Q}_{\tilde{j},\tilde{k},m} \rangle\rangle_L}{k n_{\tilde{j},1} c_{i_{\tilde{j},1}} \pi \langle\langle \tilde{Q}_{\tilde{j},\tilde{k},m}, \tilde{Q}_{\tilde{j},\tilde{k},m} \rangle\rangle_L}$$

for

$$F(x) := [F^1(l_1 x/\pi), F^2(l_2 x/\pi), \ldots, F^N(l_N x/\pi)]^{\mathrm{T}},$$

$$G(x) := [G^1(l_1 x/\pi), G(l_2 x/\pi), \ldots, G^N(l_N x/\pi)]^{\mathrm{T}}.$$

The solution of the problem (5)–(9) is finally obtained through the substitution

$$u^i(x_i, t) = v^i(\pi x_i/l_i, t), \quad i = 1, 2, \ldots, N.$$

## 4. A PLUCKED $N$-SYMMETRIC $N$-STRING AND ITS ENERGY

In this section, we will apply the results of the previous section in order to characterize the standing perpendicular waves of a plucked $N$-symmetric $N$-string, i.e., an $N$-string which is such that $l_1 = l_2 = \cdots = l_N =: l$, $c_1 = c_2 = \cdots = c_N =: c$ and $\rho_1 = \rho_2 = \cdots = \rho_N =: \rho$. These conditions on the $c_i$ and $\rho_i$ imply that the $N$ strings of the $N$-string form equal angles of $2\pi/N$ at their junction point, when the $N$-string is at rest (see Figure 2).

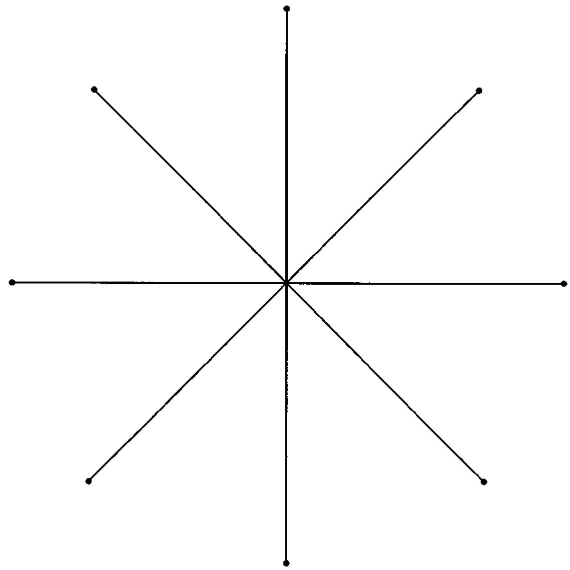

Figure 2. Schematic of a 8-symmetric 8-string.

We will suppose that our $N$-symmetric $N$-string vibrates following an initial displacement corresponding to the plucking of the string $i = N$ to the height $h$ at $x_N = l/m$, where $m \in \mathbf{R}$, $m > 1$. The factor $1/m$ indicates where the plucking is done along the string $i = N$. When $m \to \infty$, the plucking is done at the junction point of the $N$ strings. The functions which specify the initial conditions of the $N$-string are $G^i(x_i) = 0$ for $0 \leqslant x_i \leqslant l$ and $i = 1, 2, \ldots, N$, and

$$F^j(x_j) := s\left(1 - \frac{x_j}{l}\right), \quad 0 \leqslant x_j \leqslant l, \quad j = 1, 2, \ldots, N-1,$$

$$F^N(x_N) := \begin{cases} \dfrac{x_N}{l}(h-s)m + s & \text{if } 0 \leqslant x_N \leqslant l/m, \\ \dfrac{mh}{m-1}\left(1 - \dfrac{x_N}{l}\right) & \text{if } l/m < x_N \leqslant l, \end{cases}$$

where

$$s := \frac{mh}{N+m-1}.$$

Note that in this case, there is one and only one resonance which is

$$\frac{c_1}{l_1} = \frac{c_2}{l_2} = \cdots = \frac{c_N}{l_N}.$$

The solutions of equation (24) and the corresponding eigenfunctions are given by

$$\lambda_k = \alpha_k = \left(k - \frac{1}{2}\right)\pi c/l, \quad \lambda_k = k\pi c/l, \quad k \in \mathbf{N}^*,$$

and

$$\begin{aligned} P_k(x) &= [1, 1, \ldots, 1]^{\mathrm{T}} \cos(k - \tfrac{1}{2})x, \\ Q_{1,k,1}(x) &= [(N-1), -1, -1, \ldots, -1]^{\mathrm{T}} \sin kx, \\ Q_{1,k,2}(x) &= [0, (N-2), -1, -1, \ldots, -1]^{\mathrm{T}} \sin kx, \\ &\vdots \\ Q_{1,k,N-2}(x) &= [0, 0, \ldots, 2, -1, -1]^{\mathrm{T}} \sin kx, \\ Q_{1,k,N-1}(x) &= [0, 0, \ldots, 0, 1, -1]^{\mathrm{T}} \sin kx. \end{aligned}$$

Applying the results of section 3 with these eigenvalues and eigenfunctions, we find

$$\begin{aligned} u^j(x_j, t) = \frac{2hm^2}{\pi^2(m-1)(N+m-1)} \sum_{k=1}^{\infty} \Bigg[ &\frac{\cos(k-\frac{1}{2})\pi/m}{(k-\frac{1}{2})^2} \cos\frac{(k-\frac{1}{2})\pi x_j}{l} \cos\frac{(k-\frac{1}{2})\pi ct}{l} \\ &- \frac{\sin\frac{k\pi}{m}}{k^2} \sin\frac{k\pi x_j}{l} \cos\frac{k\pi ct}{l} \Bigg] \end{aligned} \tag{28}$$

for $j = 1, 2, \ldots, N-1$, and

$$u^N(x_j, t) = \frac{2hm^2}{\pi^2(m-1)(N+m-1)} \sum_{k=1}^{\infty} \left[ \frac{\cos(k-\frac{1}{2})\pi/m}{(k-\frac{1}{2})^2} \cos\frac{(k-\frac{1}{2})\pi x_N}{l} \cos\frac{(k-\frac{1}{2})\pi ct}{l} \right.$$
$$\left. + (N-1)\frac{\sin\frac{k\pi}{m}}{k^2} \sin\frac{k\pi x_N}{l} \cos\frac{k\pi ct}{l} \right]. \tag{29}$$

Expressions (28) and (29) show that the vibrations of an $N$-symmetric $N$-string can be represented in terms of the functions $u^i(x_i, t)$, $i = 1, 2, \ldots, N$, which decompose into in-phase harmonics with frequencies $f_n := nc/4l$.

We now determine the energy contained in each harmonic of the $N$-string vibrations. The energy of any one of these harmonics is the sum of its kinematic and potential energies in each of the $N$ strings of the $N$-string. The kinematic $K_n^i$ and potential $V_n^i$ energies of the $n$th harmonic of the $i$th string, $n \in \mathbf{N}^*$, $i = 1, 2, \ldots, N$, are given by

$$K_n^i = \frac{\rho}{2} \int_0^l [(u_n^i)_t(x_i, t)]^2 \, \mathrm{d}x_i$$

and

$$V_n^i = \frac{\rho c^2}{2} \int_0^l [(u_n^i)_x(x_i, t)]^2 \, \mathrm{d}x_i.$$

The total energy $E_n(N, m, h)$ of the $n$th harmonic is thus

$$E_n(N, m, h) = \sum_{i=1}^{N} (K_n^i + V_n^i) = \begin{cases} \dfrac{4\rho c^2 h^2 m^4 N (\cos\frac{n\pi}{2m})^2}{\pi^2 l n^2 (m-1)^2 (N+m-1)^2} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{4\rho c^2 h^2 m^4 N (N-1) (\sin\frac{n\pi}{2m})^2}{\pi^2 l n^2 (m-1)^2 (N+m-1)^2} & \text{if } n \text{ is even.} \end{cases}$$

Now, since we have conservation of energy, the total energy $E(N, m, h)$ of all the harmonics of the $N$-string is the same as its initial energy, which is the potential energy associated with its initial displacement. We easily obtain that

$$E(N, m, h) = \frac{\rho c^2 N m^2 h^2}{2l(m-1)(N+m-1)}.$$

The fraction of the total energy in the $n$th harmonic is therefore given by

$$R(N, m, n) := \frac{E_n(N, m, h)}{E(N, m, h)} = \begin{cases} \dfrac{8m^2 (\cos\frac{n\pi}{2m})^2}{\pi^2 n^2 (m-1)(N+m-1)} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{8m^2 (N-1) (\sin\frac{n\pi}{2m})^2}{\pi^2 n^2 (m-1)(N+m-1)} & \text{if } n \text{ is even.} \end{cases} \tag{30}$$

From (30), one immediately obtains

$$\lim_{m \to \infty} R(N, m, n) = \begin{cases} \dfrac{8}{\pi^2 n^2} & \text{if } n \text{ is odd,} \\[2ex] 0 & \text{if } n \text{ is even.} \end{cases}$$

and

$$R(m, n) := \lim_{N \to \infty} R(N, m, n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{8m^2 (\sin \frac{n\pi}{2m})^2}{\pi^2 n^2 (m-1)} & \text{if } n \text{ is even.} \end{cases} \tag{31}$$

Expression (30) shows that $R(N, m, 2k-1) = 0$ if $m$ divides $2k-1$, and $R(N, m, 2k) = 0$ if $m$ divides $k$. Using equation (30), it is moreover easy to show that $R(N, m, 2) \geqslant R(N, m, 1)$ for $1 < m \leqslant \pi/\arccos((N-3)/(N-1))$ and $N \in \mathbf{N}$, $N > 2$. Similar inequalities can be derived for other harmonics. Thus, in contrast to an ordinary string (i.e., $N = 2$) where the first harmonic is always the most energetic wherever the string is plucked, for an $N$-symmetric $N$-string with $N > 2$, it is possible to excite higher harmonics to an energy level above that of the first harmonic simply by plucking the string at an appropriate value of $m$. Expression (30) also shows that when an $N$-symmetric $N$-string is plucked at the junction point of its strings, the energy levels of its higher modes are in the same harmonic relation, with respect to the energy level of its fundamental mode, as in the case of an ordinary string.

The graphs of Figure 3 give $R(N, m, n)$ in terms of $m$ for $N = 2, 3, 5, 10$ and $n = 1, 2, 3, 4$. The graphs of Figure 4 give $R(N, m, n)$ in terms of $n$ for $N = 2, 3, 5, 10$ and $m = 2, 3, 4, 5$, showing the $\approx 1/n^2$ decay of the mode energies. Finally, from equation (31), it is easy to see that $R(m, n)$ is asymptotically given by $2/(m-1)$ for all even $n$, when $m$ is large. Figure 5 gives $R(m, n)$ in terms of $m$ for $n = 2, 4, 6, 8, 10$.

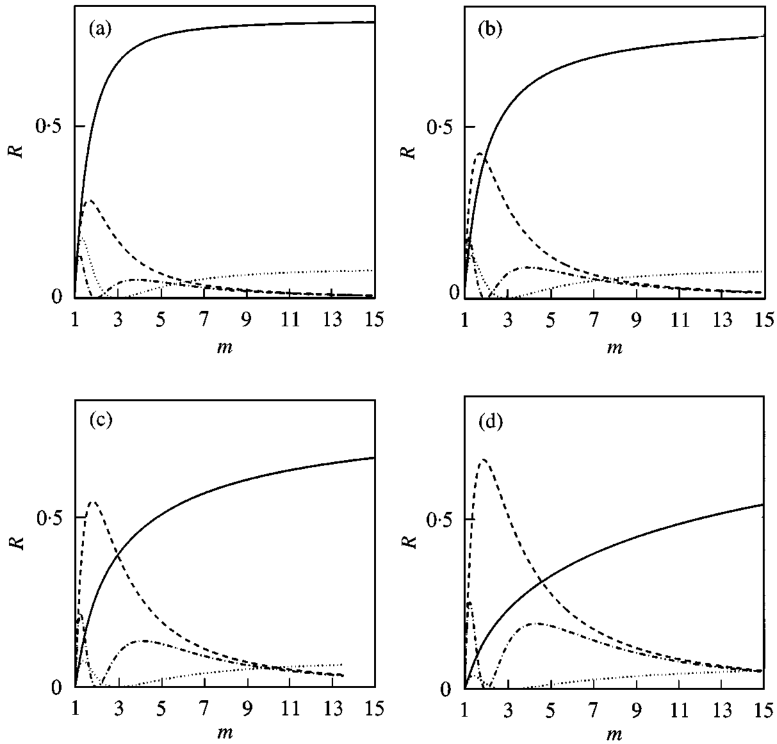


Figure 3. $R(N, m, n)$ in terms of $m$ for (*a*) $N = 2$; (*b*) 3; (*c*) 5; (*d*) 10 and ——, $n = 1$; – – –, 2; · · · ·, 3; – · –, 4.

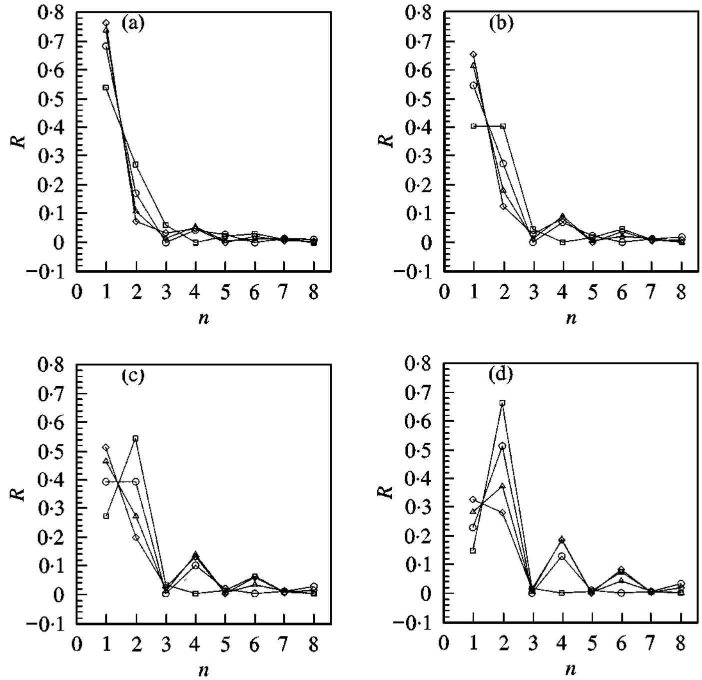


Figure 4. $R(N, m, n)$ in terms of $n$ for (*a*) $N = 2$; (*b*) 3; (*c*) 5; (*d*) 10 and □, $m = 2$; ○, 3; △, 4; ◇, 5.

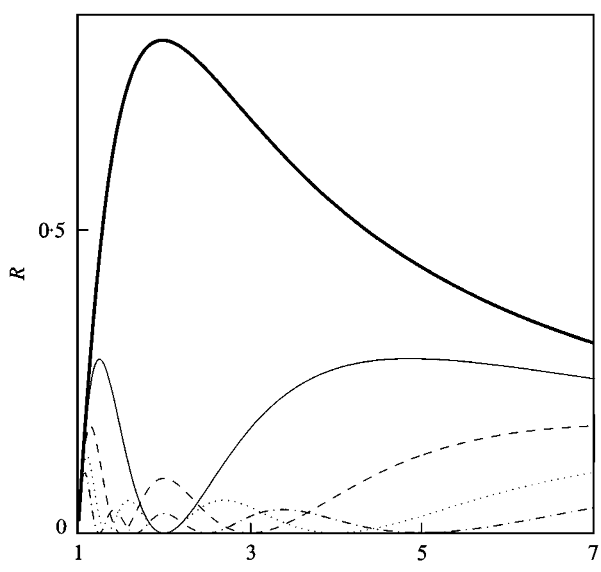


Figure 5. $R(m, n)$ in terms of $m$ for ——, $n = 2$; ——, 4; – – –, 6; …, 8; – · –, 10.

## 5. THE $(N - 1)$-SYMMETRIC $N$-STRING

We will now characterize the standing perpendicular waves of an $N$-string such that $l_2 = l_3 = \cdots = l_N =: l$, $c_1 = c_2 = \cdots = c_N =: c$, and $\rho_1 = \rho_2 = \cdots = \rho_N =: \rho$. These

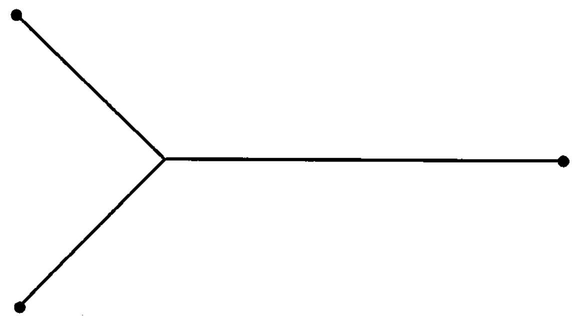

Figure 6. Schematic of a 2-symmetric 3-string.

conditions on the $c_i$ and $\rho_i$ imply that the $N$ strings of the $N$-string form equal angles of $2\pi/N$ at their point of junction, when the $N$-string is at rest. The length $l_1$ of the string $i = 1$ will, in general, be different from $l$. Such an $N$-string having one of its strings not of the same length as the $N - 1$ others will be said $(N - 1)$-symmetric (see Figure 6).

If $l_1/l$ is an irrational number, then we have the eigenfunctions $P_k(x)$ corresponding to the eigenvalues $\lambda_k = \alpha_k$ which are the positive roots of

$$(N-1)\cot\left(\frac{l\lambda}{c}\right) + \cot\left(\frac{l_1\lambda}{c}\right) = 0.$$

In this case, we also have only one resonance given by

$$\frac{c_2}{l_2} = \frac{c_3}{l_3} = \cdots = \frac{c_N}{l_N}.$$

The associated eigenvalues and eigenfunctions are respectively $\lambda_k = k\pi c/l$, $k \in \mathbf{N}^*$, and

$$Q_{1,k,1}(x) = [0, (N-2), -1, \ldots, -1]^{\mathrm{T}} \sin kx,$$

$$Q_{1,k,2}(x) = [0, 0, (N-3), -1, \ldots, -1]^{\mathrm{T}} \sin kx,$$

$$\vdots$$

$$Q_{1,k,N-3}(x) = [0, 0, \ldots, 2, -1, -1]^{\mathrm{T}} \sin kx,$$

$$Q_{1,k,N-2}(x) = [0, 0, \ldots, 0, 1, -1]^{\mathrm{T}} \sin kx.$$

If $l_1/l = p/q$, where $p$ and $q$ are relatively prime integers, we have the eigenfunctions $P_k(x)$ associated with the eigenvalues $\lambda_k = \alpha_k$ which are the positive roots of

$$(N-1)\cot\left(\frac{l\lambda}{c}\right) + \cot\left(\frac{lp\lambda}{qc}\right) = 0. \tag{32}$$

We also have the maximal resonance

$$\frac{pc_1}{l_1} = \frac{qc_2}{l_2} = \cdots = \frac{qc_N}{l_N} \tag{33}$$

and the associated eigenvalues $\lambda_k = kp\pi c/l_1$, $k \in \mathbf{N}^*$, and eigenfunctions

$$Q_{1,k,1}(x) = [(N-1)\sin kpx, -\sin kqx, \dots, -\sin kqx]^{\mathrm{T}},$$

$$Q_{1,k,2}(x) = [0, (N-2), -1, \dots, -1]^{\mathrm{T}} \sin kqx,$$

$$\vdots$$

$$Q_{1,k,N-2}(x) = [0, 0, \dots, 2, -1, -1]^{\mathrm{T}} \sin kqx,$$

$$Q_{1,k,N-1}(x) = [0, 0, \dots, 0, 1, -1]^{\mathrm{T}} \sin kqx.$$

Finally, we have a subresonance of order $q$ of equation (33) given by

$$\frac{c_2}{l_2} = \frac{c_3}{l_3} = \cdots = \frac{c_N}{l_N},$$

with associated eigenvalues $\lambda_k = \tilde{k}\pi c/l$, where $\tilde{k} \in \mathbf{N}^*$ and is not a multiple of $q$; the associated eigenfunctions are

$$\tilde{Q}_{2,\tilde{k},1}(x) = [0, (N-2), -1, -1, \dots, -1]^{\mathrm{T}} \sin \tilde{k}x,$$

$$\tilde{Q}_{2,\tilde{k},2}(x) = [0, 0, (N-3), -1, \dots, -1]^{\mathrm{T}} \sin \tilde{k}x,$$

$$\vdots$$

$$\tilde{Q}_{2,\tilde{k},N-3}(x) = [0, 0, \dots, 2, -1, -1]^{\mathrm{T}} \sin \tilde{k}x,$$

$$\tilde{Q}_{2,\tilde{k},N-2}(x) = [0, 0, \dots, 0, 1, -1]^{\mathrm{T}} \sin \tilde{k}x.$$

If $l_1 \neq l$, then there are no closed-form expressions for the roots of equation (32). In order to facilitate the solution of equation (32), one can temporarily fix the units of length and time such that $l = 1$ and $c = 1$, thus obtaining

$$(N-1)\cot\lambda + \cot\frac{p\lambda}{q} = 0. \tag{34}$$

The roots $\lambda_k = \alpha_k$ of equation (32) are then those of equation (34) multiplied by $c$ and divided by $l$. Moreover, since the functions $\cot\lambda$ and $\cot(p\lambda/q)$ are periodic respectively of period $\pi$ and $q\pi/p$, it is sufficient to determine the roots of equation (34) in the interval $[0, q\pi]$.

To be more specific let us consider in detail the case where $N = 3$ and $l_1/l = 1/2$. It is easy to show that in this particular case, the roots of equation (34) are given by $\alpha_k = \beta_k c/l$, where $\beta_k := 2[k/2]\pi + (-1)^{k-1} 2 \arctan\sqrt{2}$ and $[k/2]$ designates the integer part of $k/2$. The corresponding eigenfunctions are

$$P_k(x) = \Bigg[\cos\beta_k x/2\pi + (-1)^k \frac{\sqrt{2}}{2}\sin\beta_k x/2\pi,\ \cos\beta_k x/\pi + (-1)^{k-1}\frac{\sqrt{2}}{4}\sin\beta_k x/\pi,$$

$$\cos\beta_k x/\pi + (-1)^{k-1}\frac{\sqrt{2}}{4}\sin\beta_k x/\pi\Bigg]^{\mathrm{T}}.$$

The eigenfunctions associated with the resonances of the 3-string are respectively given by

$$Q_{1,k,1}(x) = [2\sin kx, -\sin 2kx, -\sin 2kx]^{\mathrm{T}},$$
$$Q_{1,k,2}(x) = [0, \sin 2kx, -\sin 2kx]^{\mathrm{T}}$$

and

$$\tilde{Q}_{2,\tilde{k},1}(x) = [0, \sin \tilde{k}x, -\sin \tilde{k}x]^{\mathrm{T}},$$

where $\tilde{k}$ is an odd positive integer.

The general solution to the vibrating 2-symmetric 3-string problem is then given (in normalized coordinates) by

$$\begin{aligned}
&[v^1(x,t), v^2(x,t), v^3(x,t)]^{\mathrm{T}} \\
&\quad = \sum_{k=1}^{\infty} \Bigg[ (a_k \cos \beta_k ct/l + \hat{a}_k \sin \beta_k ct/l) P_k(x) \\
&\qquad + \sum_{m=1}^{2} (b_{1,k,m} \cos 2k\pi ct/l + \hat{b}_{1,k,m} \sin 2k\pi ct/l) Q_{1,k,m}(x) \\
&\qquad + (c_{2,2k-1,1} \cos(2k-1)\pi ct/l + \hat{c}_{2,2k-1,1} \sin(2k-1)\pi ct/l) \tilde{Q}_{2,2k-1,1}(x) \Bigg],
\end{aligned} \tag{35}$$

where the coefficients $a_k$, $\hat{a}_k$ and so forth are determined by the initial conditions as indicated in section 3.2. The solution of equations (5)–(9) is obtained through the substitution

$$u^1(x_1,t) = v^1(\pi x_1/l_1, t), \quad u^2(x_2,t) = v^2(\pi x_2/l, t), \quad u^3(x_3,t) = v^3(\pi x_3/l_3, t).$$

Expression (35) shows that the vibrations of our particular 2-symmetric 3-string can be represented in terms of the functions $u^i(x_i,t)$, $i = 1, 2, 3$, which decompose into one set of in-phase harmonics with frequencies $f_n = nc/2l$, $n \in \mathbf{N}^*$, and one set of modes of frequencies $\tilde{f}_n = (2[n/2]\pi + (-1)^{n-1} 2\arctan\sqrt{2})c/2\pi l$, $n \in \mathbf{N}^*$. Note that $\tilde{f}_n$ is not an integer multiple of $\tilde{f}_1$, which is the lowest frequency of this kind. The vibrations of our 2-symmetric 3-string may thus be expressed in terms of two infinite sets of periodic functions, but which are not of the same period for each value of $n$: their frequencies are in fact incommensurate. Consequently, the general solution of our 2-symmetric 3-string will not, in general, be periodic in time. The form of equation (34) implies that the preceding observation applies to all $(N-1)$-symmetric $N$-string, when $l_1 \neq l$.

Let us now determine the proportion of energy which is devoted to the harmonic vibrations over the total energy of all vibrations resulting from the plucking of one string of our 2-symmetric 3-string. To this end, we will admit that the 3-string vibrates following an initial displacement corresponding to the plucking of the string $i = 3$ to the height $h$ at the point $x_3 = l/m$, where $m \in \mathbf{R}$, $m > 1$. The functions specifying the initial conditions of the 3-string are $G^1(x_1) = 0$ for $0 \leqslant x_1 \leqslant l/2$, $G^i(x_i) = 0$ for $0 \leqslant x_i \leqslant l$ and $i = 2, 3$, and

$$F^1(x_1) := \frac{mh}{m+3}\left(1 - \frac{2x_1}{l}\right), \quad 0 \leqslant x_1 \leqslant l/2,$$

$$F^2(x_2) := \frac{mh}{m+3}\left(1 - \frac{x_2}{l}\right), \quad 0 \leqslant x_2 \leqslant l,$$

$$F^3(x_3) := \begin{cases} \frac{mh}{m+3}\left(1 + \frac{3x_3}{l}\right) & \text{if } 0 \leqslant x_3 \leqslant l/m \\ \frac{mh}{m-1}\left(1 - \frac{x_3}{l}\right) & \text{if } l/m < x_3 \leqslant l. \end{cases}$$

It follows that

$$u^1(x_1,t)=\sum_{k=1}^{\infty}\left[a_k\cos\frac{\beta_k ct}{l}\left(\cos\frac{\beta_k x_1}{l}+(-1)^k\frac{\sqrt{2}}{2}\sin\frac{\beta_k x_1}{l}\right)\right.$$
$$\left.+\,2b_{1,k,1}\cos\frac{2k\pi ct}{l}\sin\frac{2k\pi x_1}{l}\right],\tag{36}$$

$$u^2(x_2,t)=\sum_{k=1}^{\infty}\left[a_k\cos\frac{\beta_k ct}{l}\left(\cos\frac{\beta_k x_2}{l}+(-1)^{k-1}\frac{\sqrt{2}}{4}\sin\frac{\beta_k x_2}{l}\right)\right.$$
$$-\,(b_{1,k,1}-b_{1,k,2})\cos\frac{2k\pi ct}{l}\sin\frac{2k\pi x_2}{l}$$
$$\left.+\,c_{2,2k-1,1}\cos\frac{(2k-1)\pi ct}{l}\sin\frac{(2k-1)\pi x_2}{l}\right],\tag{37}$$

$$u^3(x_3,t)=\sum_{k=1}^{\infty}\left[a_k\cos\frac{\beta_k ct}{l}\left(\cos\frac{\beta_k x_3}{l}+(-1)^{k-1}\frac{\sqrt{2}}{4}\sin\frac{\beta_k x_3}{l}\right)\right.$$
$$-\,(b_{1,k,1}+b_{1,k,2})\cos\frac{2k\pi ct}{l}\sin\frac{2k\pi x_3}{l}$$
$$\left.-\,c_{2,2k-1,1}\cos\frac{(2k-1)\pi ct}{l}\sin\frac{(2k-1)\pi x_3}{l}\right],\tag{38}$$

where the coefficients $a_k$, $b_{1,k,1}$, $b_{1,k,2}$, and $c_{2,2k-1,1}$ are determined by the initial conditions.

The energy of the harmonic vibrations of our plucked 2-symmetric 3-string is the sum of the kinematic and potential energies of the harmonics $\cos n\pi ct/l$, $\sin n\pi x_i/l$, $n\in\mathbf{N}^*$, $i=1,2,3$, in equations (36)–(38). One easily shows that the ratio $R(m)$ of this energy over the total energy of the plucked 3-string is given by

$$R(m)=\frac{2m^2}{\pi^2(m-1)(m+3)}\sum_{k=1}^{\infty}\left[\frac{2(\sin\frac{(2k-1)\pi}{m})^2}{(2k-1)^2}+\frac{3(\sin\frac{2k\pi}{m})^2}{(2k)^2}\right].\tag{39}$$

The graph of Figure 7 represents the function $R(m)$ for $1<m\leqslant 10$. From equation (39), or directly from the expressions of $b_{1,k,1}$, $b_{1,k,2}$ and $c_{2,2k-1,1}$, it is possible to show that $R(m)\to 0$ when $m\to\infty$.

## 6. TRAVELLING WAVES

The model considered in this paper remains valid when all strings of the $N$-string are of infinite length. In this section and the next, we will consider solutions (possibly weak solutions) of equations (5), (6) and (8) which represent perpendicular travelling waves. Since $0\leqslant x_i<\infty$ for $i=1,2,\ldots,N$, we will replace each $x_i$ by $0\leqslant x<\infty$.

Let us suppose a travelling perpendicular wave on the first string of the $N$-string which approaches the junction point $O$ with speed $c_1$ and profile $f_1:\mathbf{R}\to\mathbf{R}$. This travelling wave can be described as

$$u^1(x,t)=f_1(t+x/c_1).$$

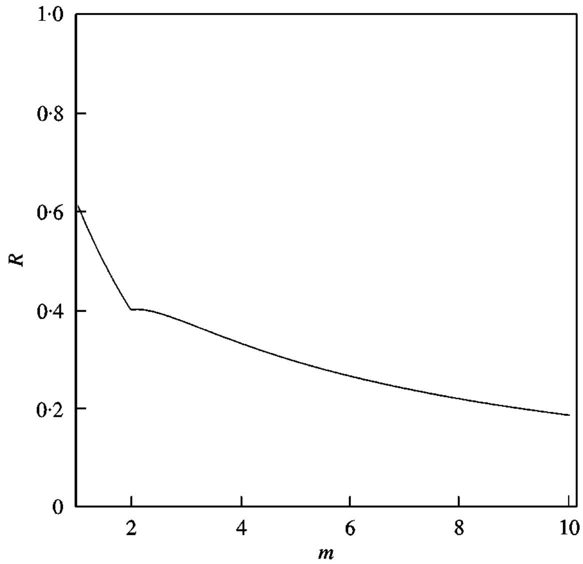


Figure 7. $R(m)$ in terms of $m$.

If a part of the wave is reflected by the point $O$ then this reflected wave will travel in the positive $x_1$ direction and can be described by $u^1(x, t) = g_1(t - x/c_1)$, where $g_1 : \mathbf{R} \to \mathbf{R}$. The total perpendicular displacement of points along the first string is then given by

$$u^1(x, t) = f_1(t + x/c_1) + g_1(t - x/c_1).$$

The part of the incident wave at $O$ that is transmitted in the other $N - 1$ strings will also form travelling perpendicular waves which can be described as

$$u^j(x, t) = h_j(t - x/c_j), \quad j = 2, 3, \ldots, N,$$

where $h_j : \mathbf{R} \to \mathbf{R}$. Conditions (6) then imply

$$f_1(t) + g_1(t) = h_j(t), \quad j = 2, 3, \ldots, N. \tag{40}$$

At the instant when the travelling perpendicular wave reaches point $O$ along the first string, equation (8) is verified. This implies that

$$f_1'(t) - g_1'(t) = \frac{1}{v_1} \sum_{j=2}^{N} v_j h_j'(t). \tag{41}$$

Integration of equation (41) yields

$$f_1(t) - g_1(t) = \frac{1}{v_1} \sum_{j=2}^{N} v_j h_j(t) + K_1, \tag{42}$$

where $K_1$ is an arbitrary constant. Let us suppose that there exists a time $t_0$ such that $f_1(t_0) = g_1(t_0) = h_j(t_0) = 0$, for $j = 2, 3, \ldots, N$. This leads to $K_1 = 0$ and

$$f_1(t) - g_1(t) = \frac{1}{v_1} \sum_{j=2}^{N} v_j h_j(t). \tag{43}$$

From equations (40) and (43), we get

$$g_1(t) = \left(\frac{2v_1}{NA} - 1\right) f_1(t),$$

$$h_j(t) = \frac{2v_1}{NA} f_1(t), \quad j = 2, 3, \ldots, N,$$

where $A := (1/N) \sum_{i=1}^{N} v_i$ is the arithmetic mean of the $v_i$.

Since all the $v_i$ are finite numbers, the value of $A$ is finite for every $N \in \mathbf{N}, N \geqslant 2$. Therefore $g_1(t) \to -f_1(t)$ and $h_j(t) \to 0$, for $j = 2, 3, \ldots, N$, when $N \to \infty$.

If the expression $f_1(t + x/c_1)$ of the incident wave at $O$ is known, the reflected and transmitted waves by the point $O$ are given by

$$g_1(t - x/c_1) = (\tfrac{2v_1}{NA} - 1) f_1(t - x/c_1),$$

$$h_j(t - x/c_j) = \tfrac{2v_1}{NA} f_1(t - x/c_j), \quad j = 2, 3, \ldots, N$$

respectively. In particular, if the incident wave at $O$ is harmonic with frequency $n$, i.e., if

$$f_1(t - x/c_1) = A_1 \exp[2\pi \mathrm{i}(nt - k_1 x)],$$

where $A_1$ represents the amplitude and $k_1 = n/v_1$, it is easy to see that the reflected and transmitted waves are harmonic as well and have the same frequency as the incident wave.

## 7. TRAVELLING WAVE INTERACTIONS

Let us consider $N - 1$ travelling perpendicular waves which reach $O$ at the same time, one along each of the first $N - 1$ strings of the $N$-string. Using the method presented above one directly shows that the perpendicular displacement of points on these $N - 1$ strings are

$$u^k(x, t) = f_k(t + x/c_k) + g_k(t - x/c_k), \quad k = 1, 2, \ldots, N - 1,$$

where the $f_k$, and $g_k$, $k = 1, 2, \ldots, N - 1$, are functions of $\mathbf{R}$ in $\mathbf{R}$ which describe the incident and the corresponding reflected waves, respectively. The transmitted wave in the $N$th string of the $N$-string is then described by

$$u^N(x, t) = h_N(t - x/c_N),$$

where $h_N : \mathbf{R} \to \mathbf{R}$.

Conditions (40) correspond here to

$$f_k(t) + g_k(t) = h_N(t), \quad k = 1, 2, \ldots, N - 1.$$

Using equation (8), one easily shows that the equation analogous to equation (42) is

$$\sum_{k=1}^{N-1} v_k [f_k(t) - g_k(t)] = v_N h_N(t) + K_2,$$

where $K_2$ is an arbitrary constant. Here again we suppose that there exists a time $\tau_0$ such that $f_k(\tau_0) = g_k(\tau_0) = 0$, for $k = 1, 2, \ldots, N-1$ and $h_N(\tau_0) = 0$. It follows that $K_2 = 0$ and

$$g_k(t) = \frac{2}{NA} \sum_{j=1}^{N-1} v_j f_j(t) - f_k(t), \quad k = 1, 2, \ldots, N-1,$$

$$h_N(t) = \frac{2}{NA} \sum_{j=1}^{N-1} v_j f_j(t).$$

Let $c_1 = c_2 = \cdots = c_N$ and $\rho_1 = \rho_2 = \cdots = \rho_N$, which imply that the $N$ strings of the $N$-string at rest form equal angles of $2\pi/N$. Now if $f_1(t) = f_2(t) = \cdots = f_{N-1}(t)$, then $g_k(t) = (1 - 2/N) f_1(t)$ for $k = 1, 2, \ldots, N-1$, and $h_N(t) = 2(1 - 1/N) f_1(t)$. Given that the energy per unit length of a travelling wave in a single string is proportional to the square of its amplitude, we have that the energy of the transmitted wave is equal to $4(1 - 1/N)^2$ of the energy contained in each of the $N-1$ incident waves at $O$. In this case we also have that $g_k(t) \to f_1(t)$, for $k = 1, 2, \ldots, N-1$, and $h_N(t) \to 2f_1(t)$ when $N \to \infty$.

A succession of several interactions of waves as described above could potentially produce a cascade of waves having the form of a tree whose branches represent waves having higher and higher levels of energy as they approach the trunk of the tree. To see this, let us consider a tree made up of $N$-strings with $(N-1)^M$ incident waves, $M \in \mathbf{N}^*$, at $(N-1)^{M-1}$ junction points which eventually lead to a single transmitted wave. The ratio of the amplitude of the wave exiting the tree network to that of the amplitude of the $(N-1)^M$ entering waves is given by $2^M(1 - 1/N)^M$ and hence their energy ratio is $2^{2M}(1 - 1/N)^{2M}$. The ratio between the transmitted wave exiting the tree network and the total energy of the entering waves is $(N-1)^M(2/N)^{2M}$. Such a device could be used as a means of amplifying a travelling wave, provided there is a way of reproducing it. Copies of this wave would have to be diffused in wave guides configured as a tree network. The inverse process could be used to reduce the amplitude of a travelling wave. Note that these observations are only relevant if the amplitudes of all travelling waves in the tree network are sufficiently small for our linear mathematical model to be applicable.

## 8. CONCLUSION AND DISCUSSION

We have studied a linear model for the transverse vibrations of an $N$-string. This model is useful for describing the small-amplitude oscillations of the $N$-string. Also, our analysis could help to shed some light on any nonlinear model whose linearization about the rest state reduces to our model (i.e., characterizing the linear stability of this rest state).

The general solution for standing perpendicular waves presented in Section 3 applies even when the $N$ strings of the $N$-string at rest do not form equal angles at their junction point. If these angles are not equal, the solution takes into account the corresponding different tensions in the strings through the values of the constants $c_i$, $i = 1, 2, \ldots, N$.

For a plucked $N$-symmetric $N$-string with $N > 2$, we have found that higher harmonics can be excited to an energy level above that of the first harmonic simply by plucking at an appropriate place along one of the strings. This phenomenon does not occur for an ordinary plucked string. Acoustically, this result could lead to a unique tone colour which could lead to the design of new musical instruments. A physical model of an $N$-string is being built to explore this further. A more complete model of the $N$-string incorporating effects such as air resistance will also be investigated.

For an $(N-1)$-symmetric $N$-string, we have shown that, in general, the vibrations are formed of two sets of modes which, taken altogether, are not periodic in time, even if the numbers $c_i/l_i$, $i = 1, 2, \ldots, N$, are commensurate.

Finally, we have studied the propagation and interactions of travelling waves in an $N$-string with strings of infinite length. A method for increasing or reducing the amplitude of travelling prependicular waves corresponding to small deformations of an $N$-string has also been presented.

The planar vibrations of an $N$-string were not examined in this paper. Although our mathematical model applies to this case, the situation is much more difficult and will be reported on in subsequent work.

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## REFERENCES

1. E. Butkov 1968 *Mathematical Physics*. Reading, MA: Addison-Wesley.
2. M. J. Crocker 1998 *Handbook of Acoustics*. Toronto: Wiley.
3. R. Dennemeyer 1968 *Introduction to Partial Differential Equations and Boundary Value Problems*. New York: McGraw-Hill.
4. J. Guckenheimer and P. Holmes 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer-Verlag.
5. E. J. P. G. Schmidt 1992 *SIAM Journal of Control Optimization* **30**, 229–245.
6. G. Birkhoff and G.-C. Rota 1969 *Ordinary Differential Equations*. Waltham: Blaisdell.

## APPENDIX A: PROOF OF EXISTENCE OF CERTAIN EIGENFUNCTIONS

**Lemma A1.** *The operator $A$ is symmetric with respect to $\langle\langle\, , \rangle\rangle_L$ on $\mathbf{\Delta}$.*

**Proof**. Let $\Phi$, $\Psi \in \mathbf{\Delta}$. We then have

$$\begin{aligned}
\langle\langle A\Phi, \Psi\rangle\rangle_L &= \int_0^\pi \langle -C\Phi''(x), \Psi(x)\rangle_L \,\mathrm{d}x \\
&= \langle -C\Phi'(x), \Psi(x)\rangle_L|_0^\pi - \int_0^\pi \langle -C\Phi'(x), \Psi'(x)\rangle_L \,\mathrm{d}x \\
&= -\int_0^\pi \langle -C\Phi'(x), \Psi'(x)\rangle_L \,\mathrm{d}x \\
&= -\langle -C\Phi(x), \Psi'(x)\rangle_L|_0^\pi + \int_0^\pi \langle \Phi(x), -C\Psi''(x)\rangle_L \,\mathrm{d}x \\
&= \int_0^\pi \langle \Phi(x), -C\Psi''(x)\rangle_L \,\mathrm{d}x = \langle\langle \Phi, A\Psi\rangle\rangle_L. \quad \square
\end{aligned}$$

**Proposition A1.** *The spectrum of $A$ is composed only of discrete eigenvalues of multiplicity at most $2N$. The eigenvalues of $A$ constitute a real monotone sequence $\{\omega_n\}$ such that*

$$\omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_n \leqslant \omega_{n+1} \leqslant \cdots, \quad \lim_{n\to\infty} \omega_n = \infty.$$

*Furthermore, there exists an orthonormal set $\{\Phi_i\}$ of eigenfunctions of $A$ which is complete in $\mathbf{\Lambda}$.*

**Proof.** It follows from Lemma A1, and is a simple generalization of what is known for scalar Sturm–Liouville problems [6]; namely, one constructs a "Green's matrix" for the problem $A\Phi = \omega\Phi$ and an associated compact self-adjoint integral operator with kernel given by this Green's matrix, and whose range is dense in $\mathbf{\Lambda}$. □

## APPENDIX B

**Proof.** Let $\bar{\mathbf{\Sigma}} \subset \mathbf{\Lambda}$ be the closure of the linear span $\mathbf{\Sigma}$ of all eigenfunctions $Q_{j,k,m}$ and $\tilde{Q}_{j,\tilde{k},m}$ previously defined. The result will follow from Proposition A1 of Appendix A if we can show that the orthogonal complement $\bar{\mathbf{\Sigma}}^{\perp}$ is infinite dimensional. We recall the fact that if $k_1$ and $k_2$ are integers, then

$$\int_0^{\pi} \sin k_1 x \sin k_2 x \, \mathrm{d}x = \begin{cases} 0 & \text{if } k_1 \neq k_2, \\ \pi/2 & \text{if } k_1 = k_2. \end{cases}$$

Consider the first resonance in equation (27), which is necessarily maximal. For any positive integer $n$, we define the function $R_n(x) \in \mathbf{\Lambda}$ as having all its components zero except those with indices $i_{1,1}, i_{1,2}, \ldots, i_{1,p1}$, which are of the form

$$R_n^{i_{1,1}}(x) := V^{i_{1,1}} \sin n n_{1,1} x,$$

$$R_n^{i_{1,2}}(x) := V^{i_{1,2}} \sin n n_{1,2} x,$$

$$\vdots$$

$$R_n^{i_{1,p1}}(x) := V^{i_{1,p1}} \sin n n_{1,p1} x,$$

where $V^{i_{1,1}}, V^{i_{1,2}}, \ldots, V^{i_{1,p1}}$ are chosen such that

$$\frac{l_{i_{1,1}}\rho_{i_{1,1}}}{v_{i_{1,1}}} V^{i_{1,1}} \left( \frac{v_{i_{1,2}}}{n_{i_{1,2}}} + \frac{v_{i_{1,3}}}{n_{i_{1,3}}} + \cdots + \frac{v_{i_{1,p1}}}{n_{i_{1,p1}}} \right) - \frac{l_{i_{1,2}}\rho_{i_{1,2}}}{n_{i_{1,2}}} V^{i_{1,2}} - \cdots - \frac{l_{i_{1,p1}}\rho_{i_{1,p1}}}{n_{i_{1,p1}}} V^{i_{1,p1}} = 0,$$

$$\frac{l_{i_{1,2}}\rho_{i_{1,2}}}{v_{i_{1,2}}} V^{i_{1,2}} \left( \frac{v_{i_{1,3}}}{n_{i_{1,3}}} + \cdots + \frac{v_{i_{1,p1}}}{n_{i_{1,p1}}} \right) - \frac{l_{i_{1,3}}\rho_{i_{1,3}}}{n_{i_{1,3}}} V^{i_{1,3}} - \cdots - \frac{l_{i_{1,p1}}\rho_{i_{1,p1}}}{n_{i_{1,p1}}} V^{i_{1,p1}} = 0,$$

$$\vdots$$

$$\frac{l_{i_{1,p1-1}}\rho_{i_{1,p1-1}}}{v_{i_{1,p1-1}}} V^{i_{1,p1-1}} \left( \frac{v_{i_{1,p1}}}{n_{i_{1,p1-1}}} \right) - \frac{l_{i_{1,p1}}\rho_{i_{1,p1}}}{n_{i_{1,p1}}} V^{i_{1,p1}} = 0.$$

By construction, $R_n(x)$ is orthogonal to each $Q_{1,k,m}(x)$. If the $j$th resonance ($j \geqslant 2$) in equation (27) does not involve any of the $c_{i_{1,1}}/l_{i_{1,1}}, c_{i_{1,2}}/l_{i_{1,2}}, \ldots, c_{i_{1,p1}}/l_{i_{1,p1}}$, then $R_n(x)$ is also orthogonal to each $Q_{j,k,m}(x)$ or to each $\tilde{Q}_{j,\tilde{k},m}(x)$, whichever applies depending on whether or not the $j$th resonance is maximal. Finally, if the $j$th resonance ( $j \geqslant 2$) in equation (27) does involve one of the $c_{i_{1,1}}/l_{i_{1,1}}, c_{i_{1,2}}/l_{i_{1,2}}, \ldots, c_{i_{1,p1}}/l_{i_{1,p1}}$, then this resonance is a subresonance of the first resonance, with order of inclusion $K > 1$. Since the $\tilde{Q}_{j,\tilde{k},m}$ are such that $\tilde{k}$ is not a multiple of $K$, then a simple computation shows that $R_n(x)$ is orthogonal to each $\tilde{Q}_{j,\tilde{k},m}(x)$. Thus, for each integer $n$ we have that $R_n(x) \in \bar{\mathbf{\Sigma}}^{\perp}$. □